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ON BUSEMANN'S DIAGRAMS FOR STEADY (STATIONARY) SHOCKS

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ON BUSEMANN'S DIAGRAMS FOR STEADY (STATIONARY) SHOCKS.

1. Linear shock.

Throughout we shall assume that the fluid is in a steady state. Presently we assume it to fill a straight tube so that its state can be described by one velocity component u, and by the pressure p and the density ρ. We assume that pressure p and density ρ are connected through a relation

$$(1.1) \qquad p = \sigma(s)\, f(\rho)$$

s denoting the entropy of the state (*). The sound speed is then given by

$$(1.2) \qquad c = [\gamma\, p/\rho\,]^{1/2} \quad .$$

We also introduce the enthalpy i, and consider it also a function of ρ and s (⁑),

$$(1.3) \qquad i = \sigma(s)\, g(\rho) = \int^{\rho} \frac{c^2}{\rho}\, d\rho \quad .$$

The quantities p and i are related through

$$(1.4) \qquad \rho\, \partial i/\partial\rho = \partial p/\partial\rho = c^2, \text{ i.e. } \rho\, g'(\rho) = f'(\rho) \quad .$$

If, in particular, the gas is ideal,

$$(1.1) \qquad p = \sigma(s)\rho^{\gamma}, \quad 1 < \gamma,$$

we have

$$(1.2) \qquad c = [\gamma\rho^{-1}p]^{1/2}$$

and

$$(1.3) \qquad i = \frac{\gamma}{\gamma - 1}\, \rho^{-1}p = c^2/(\gamma - 1).$$

Further the "Bernoulli quantity"

(*) We assume $f(\rho) > 0$, $f'(\rho) > 0$, $f''(\rho) > 0$, $f(0) = 0$, $f(\infty) = \infty$; $\sigma(s) > 0$, $\sigma'(s) > 0$, $\sigma(-\infty) = 0$, $\sigma(\infty) = \infty$.

(⁑) We assume $g(0) = 0$; then $\rho g(\rho) > f(\rho)$ follows from (1.4). Also relation (1.4) implies $\partial i/\partial\rho > 0$.

(1.5) $$i + \frac{1}{2} u^2$$

is introduced and all states are then considered which have the same entropy $s = s^o$ and the same "Bernoulli constant" i^o :

(1.6) $$i + \frac{1}{2} u^2 = \frac{1}{2} \hat{c}^2 = i^o \quad .$$

When s^o and i^o are prescribed, to every value u of the velocity, $u \leq \hat{c}$, the quantities ρ and p are determined, since the relation (1.3) can be solved for ρ by virtue of (1.4). In particular c and p are then definite functions of u :

(1.7) $$c = c(u) = c(u,s) \quad (*)$$

(1.8) $$p = p(u) = p(u,s) \quad (**) \quad .$$

Especially, for ideal gases,

(1.7) $$c = [\frac{\gamma - 1}{2}(\hat{c}^2 - u^2)]^{1/2}$$

(1.8) $$p = \sigma^{[-1/(\gamma - 1)]}[\frac{\gamma - 1}{2} (\hat{c}^2 - u^2)]^{\gamma/(\gamma - 1)} .$$

Among these states there will be one for which the sound speed coincides with its velocity, i.e. for which $c(u) = u$. This velocity is called "critical sound speed". Evidently we have

(1.9) $$2i^o = \hat{c}^2 = \frac{\gamma + 1}{\gamma - 1} c^{*2} \quad .$$

If $u < c^*$ we also have $u < c(u)$ and the flow is called subsonic, if $u > c^*$ we also have $u > c(u)$ and the flow is called supersonic. It is clear that it will be

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(*) By virtue of the assumptions in footnote (*),p.1, we have $u\partial c/\partial u > 0$.

(**) By virtue of the assumptions in footnote (**), p.1, the function $p(u,s)$ decreases when u increases.

convenient to use c^* as a reference velocity.

We note a few more relations needed for the discussion of the function $p = p(u)$ as given by (1.8). From (1.5) we have

$$\frac{\partial i}{\partial u} = -u \,, \tag{1.10}$$

from (1.4) we then obtain

$$\frac{\partial \rho}{\partial u} = -\rho u/c^2 \,, \tag{1.11}$$

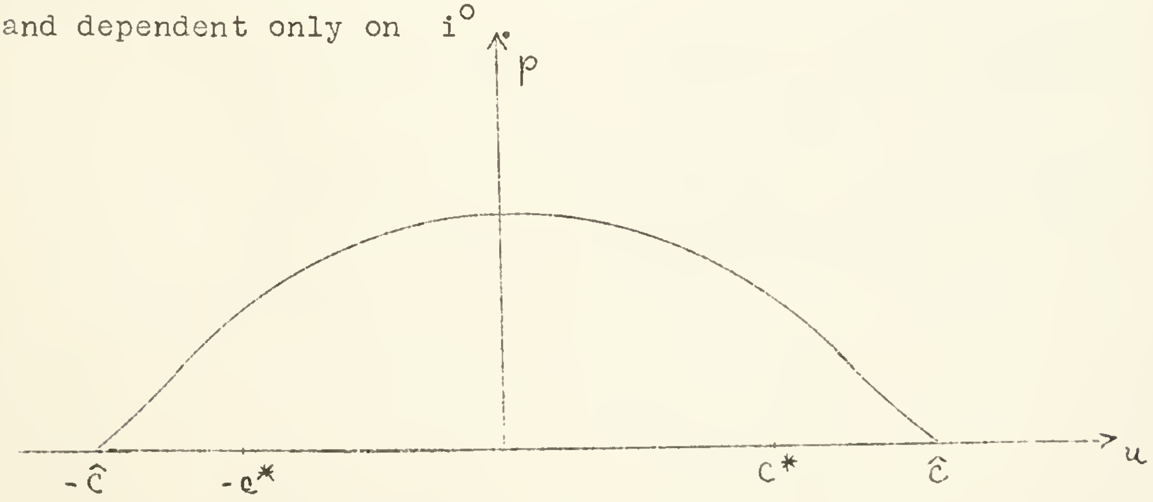
from (1.2) therefore

$$\frac{\partial p}{\partial u} = -\rho u \tag{1.12}$$

and

$$\frac{\partial^2 p}{\partial u^2} = -\rho(1 - u^2/c^2). \tag{1.13}$$

The last relation shows that the curves $p = p(u)$ have negative curvature for subsonic, positive curvature for supersonic velocity; the point of inflection is at the critical velocity. It is to be observed that separation of the two regions $u \gtrless c^*$ is independent of the entropy and dependent only on i^0

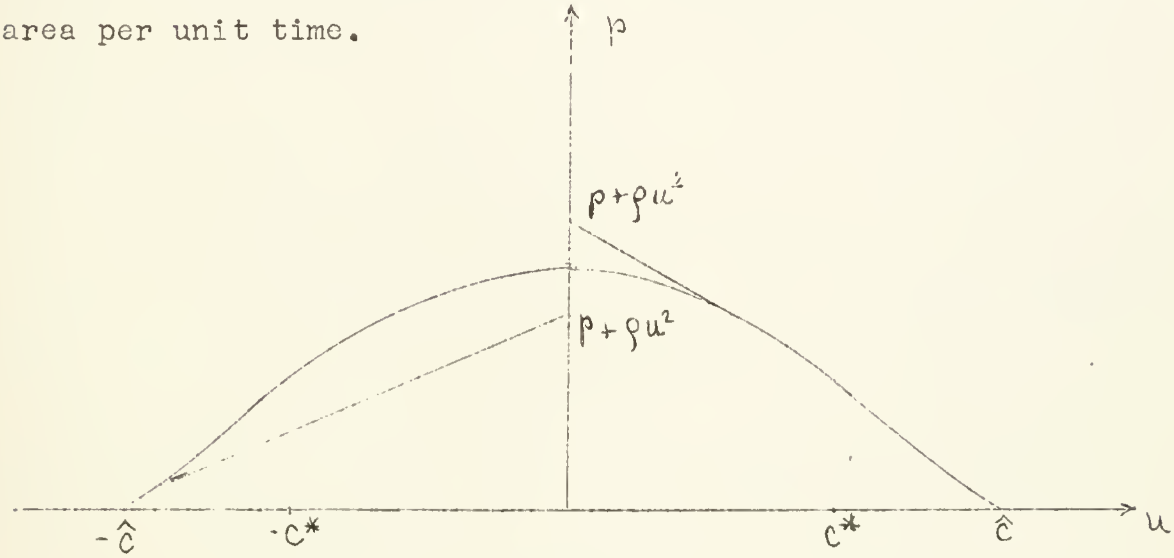


By virtue of (1.12) we can say that the negative slope of the curve $p = p(u)$ represents the mass crossing a

unit cross-section per unit time. A significant quantity is also the intercept on the p-axis of the tangent line at a point of the curve $p = p(u)$. We find for this intercept

$$p + \rho u^2 ,$$

i.e. the sum of the pressure and momentum crossing a unit area per unit time.



It should further be mentioned that the functions $p = p(u)$ for different entropies differ only by a constant factor.

It is now very easy to represent a shock transition graphically. We distinguish the left or the right hand side of the stationary shock by (χ) and (r). Assume that (χ) be the front side of the shock and that $u_\chi > 0$. The shock relations are

(1.14) $$\rho_\chi u_\chi = \rho_r u_r$$

(1.15) $$p_\chi + \rho_\chi u_\chi^2 = p_r + \rho_r u_r^2 ,$$

while the third relation can simply be written as

(1.16) $$i_r^0 = i_\chi^0 \quad \text{or} \quad c_r^* = c_\chi^* .$$

The two points in the (p,u)-plane corresponding to

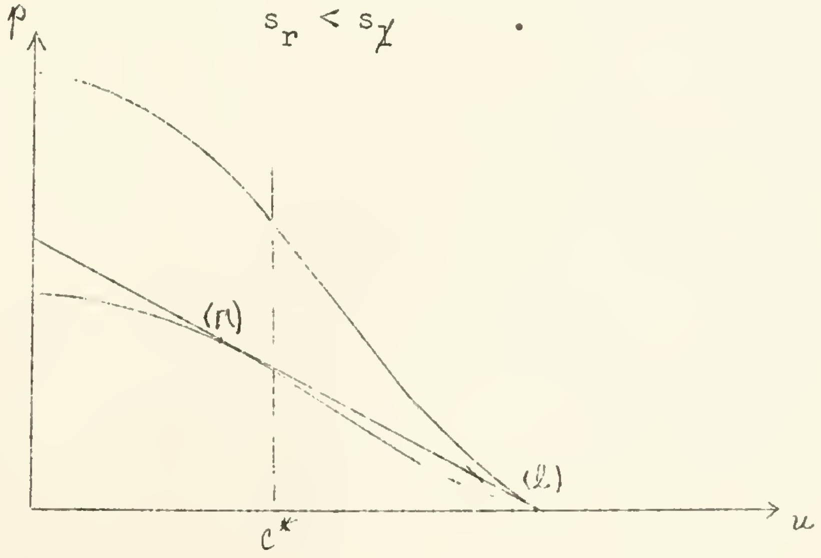
both sides of the shock may be located on two curves $p = p(u)$ having the same value c^* but differing in entropy.

The relation (1.14) can now be interpreted to say that the tangent lines at the two $p = p(u)$ curves at the points (ℓ) and (r) have the same slope, and relation (1.15) shows, in view of what was said just before, that these two tangent lines have the same intercept on the p-axis. That is, the two tangent lines coincide; or, the two points (ℓ) and (r) lie on a common tangent line for the two curves $p = p(u)$ with the same value of c^*. It is now immediately seen by use of the curvature properties of the curve $p = p(u)$ that to every point (ℓ) with $u_\ell > c^*$ there is just one point (r) having the same tangent line, clearly $u_r < c^*$. Also obviously

$$p_r(u) > p_\ell(u) \quad ,$$

which implies

$$s_r < s_\ell \quad . \tag{1.17}$$



We mention that for ideal gases the two velocities u_r and u_ℓ are related through

(1.18) $$u_\ell \cdot u_r = c^{*2} \;;$$

this follows immediately when one divides relation (1.15) by (1.14) and eliminates $\rho^{-1}p$ through (1.3), (1.6), (1.9), and (1.16) .

Another graphical representation, which perhaps may be useful, is obtained by Legendre transformation. One introduces

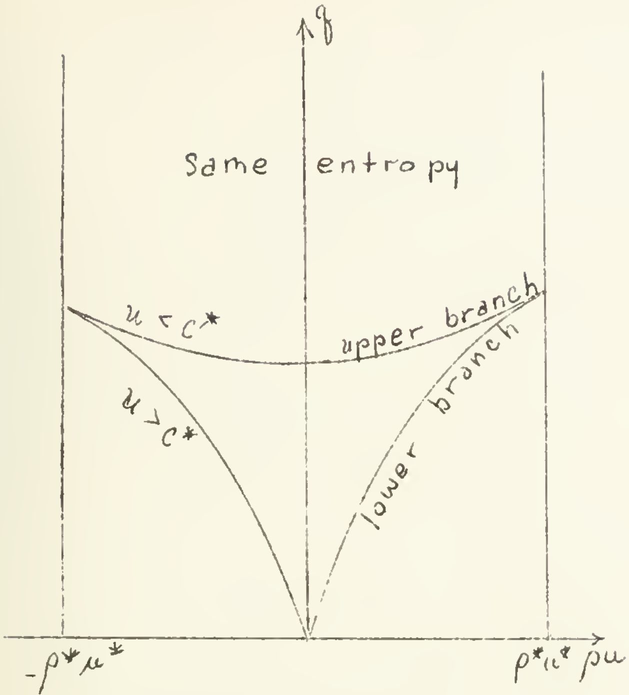
(1.19) $$q = p + \rho u^2$$

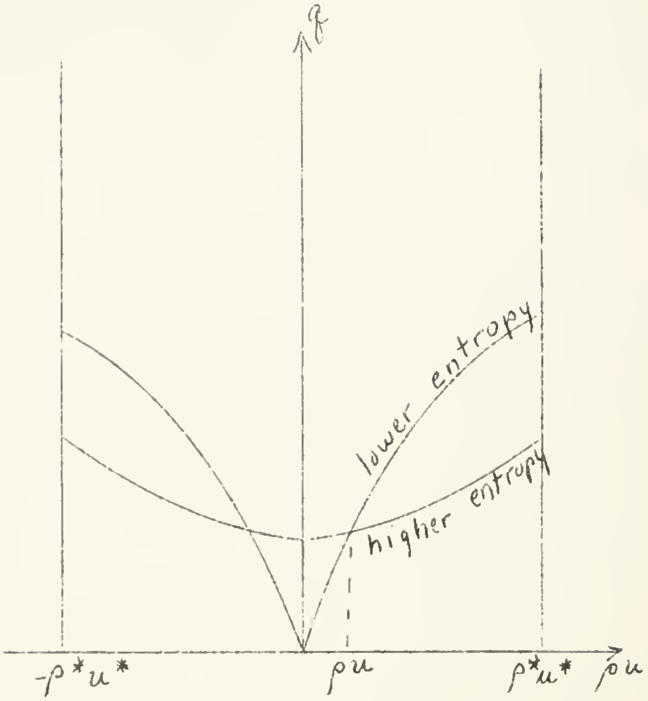
as a function of the independent variable ρu. As is seen from (1.13), this is possible for $u < c^*$ and for $u > c^*$. The domain of ρu is then

(1.20) $$0 < \rho u < \rho^* c^* \;,$$

where $\rho^* = \rho(c^*)$.

We obtain two branches for $q = q(\rho u)$, a "lower" branch, corresponding to $u > c^*$, with $q(0) = 0$, and an "upper" branch, corres ponding to $u < c^*$, with $q(0) = p^0$. For different entropies the functions $q = q(\rho u)$ differ only by a constant factor. The lower branches for different entropies do not intersect each other, and the same holds for the upper branches for different entropies. However, it is immediately seen that through every point on a lower branch one can pass an upper branch belonging to a higher entorpy. Such an intersection of lower and upper branches for different entropies corresponds to a shock transition, as is o vious from relations (1.14) and (1.15).





2. Two-dimensional shock

To investigate two-dimensional "oblique" shocks one may also consider the manifold of all states, given by velocities (u,v), by p and ρ or s, which have the same entropy s and the same Bernoulli constant

(2.1) $$i^0 = \frac{1}{2} c^2 = i + \frac{1}{2}(u^2 + v^2) \quad .$$

c and p are then functions of u and v, as a matter of fact, functions of $\sqrt{u^2 + v^2}$, which reduce for $v = 0$ to the functions of u as considered in the preceding section.

In the (u,v)-plane we consider the circles $u^2 + v^2 < c^{*2}$ consisting of subsonic states and the ring $c^{*2} < u^2 + v^2 < c^2$ containing supersonic states. Also we observe as before

(2.2) $$\frac{p}{u} = - \rho u, \qquad \frac{p}{v} = - \rho v,$$

(2.3) $$\frac{\partial^2 p}{\partial u^2} = - \rho(1 - u^2/c^2) \quad , \frac{\partial^2 p}{\partial v^2} = - \rho(1 - v^2/c^2),$$

$$\frac{\partial^2 p}{\partial u \partial v} = \frac{uv}{c^2} \quad ,$$

(2.4) $$\frac{\partial^2 p}{\partial u^2}\frac{\partial^2 p}{\partial v^2} - \left(\frac{\partial^2 p}{\partial u \partial v}\right)^2 = \rho^2\left(1 - \frac{u^2 + v^2}{c^2}\right) .$$

Thus we see that the curvature is positive or negative for subsonic or supersonic velocities, respectively. The subsonic and supersonic parts of the surface $p = p(u,v)$ may be called the "dome" and the "rim" of the surface.

Suppose now two states (χ) and (r) are separated by a stationary straight line, an "oblique" shock. Let u_n and u_t be the normal and the tangential components of the velocity. Then the transition relations are

(2.5) $$u_{\chi t} = u_{rt} ,$$

(2.6) $$\rho_\chi u_{\chi n} = \rho_r u_{rn} ,$$

(2.7) $$p_\chi + \rho_\chi u_{\chi n}^2 = p_r + \rho_r u_{rn}^2$$

and further

(2.8) $$i_\chi^0 = i_r^0 = i^0 \quad \text{or} \quad c_\chi^* = c_r^* = c^* .$$

Relation (2.5) says that the vector $(u_\chi - u_r, v_\chi - v_r)$ is normal to the shock direction, i.e. the shock direction can be found as the normal to the line connecting (u_χ, v_χ) and (u_r, v_r) in the (u,v)-plane. In the (u,v,p)-space the two points (χ) and (r) are located on two surfaces $p = p(u,v)$ having the same c^*, differing only in entropy.

It is now easily seen that the straight line in the (u,v,p)-space connecting the two points u_χ, v_χ, p_χ and u_r, v_r, p_r is a common tangent to the two surfaces $p = p(u,v)$. Indeed, the relation

(2.9) $$p_\chi - p_r + \frac{\partial p}{\partial u}\bigg|^{\chi} (u_\chi - u_r) + \frac{\partial p}{\partial v}\bigg|^{\chi} (v_\chi - v_r) = 0$$

is an immediate consequence of $\frac{\partial p}{\partial u} = - \rho u, \frac{\partial p}{\partial u} = - \rho v$, and the three shock transition relations.

We consider now one state (0) and ask for all states (1) that may be connected with (0) by an oblique shock. The straight lines in the (u,v,p)-space connecting (0) with all such states (1) will then lie on the tangent plane T_0 to $p = p(u,v)$ at the point (0). The procedure is now clear. We consider a surface $p = p(u,v)$ with any entropy and intersect it with the tangent plane T_0. In general there will be two tangent lines at the intersection curve passing through (0). The points of contact of these tangents with the intersection curve are states (1) that can be connected with (0) by a shock.

To investigate in detail the locus of the possible points (1) we first assume $u_0 > c^*$, i.e. we first assume the state u_0 to be supersonic. We then consider the surfaces $p = p(u,v)$ for decreasing entropy. When the entropy is infinite, $p \equiv 0$; when the entropy decreases the dome of the surface $p = p(u,v)$ will rise, touch, and eventually cross the tangent plane T_0 (*). The state at which the dome touches T_0 corresponds to the straight shock as considered in the preceding section. When the dome has crossed the plane T_0 the intersection will be an oval and clearly there will be two tangent lines yielding two states (1) that correspond to two possible oblique shocks. When the entropy decreases to s_0, these two points move into the point (0). So far the state (0) was on the front side, the state

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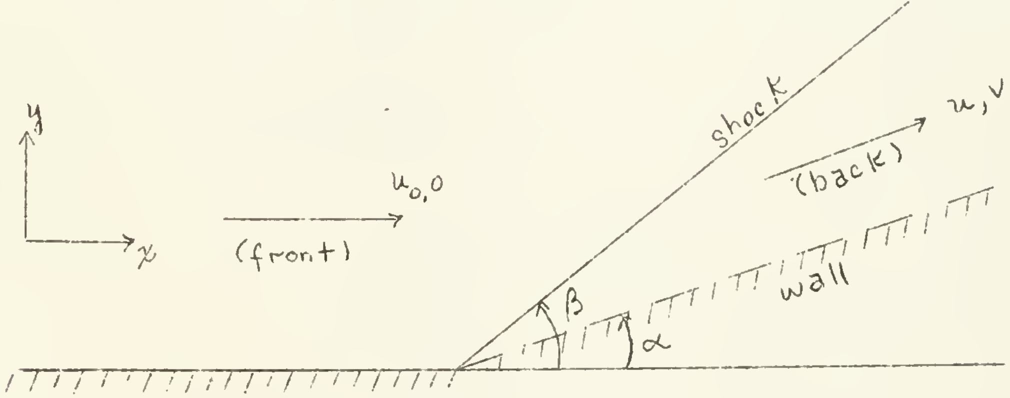
(*) It can be seen that at the intersection of the rim with the plane T_0 there are no tangent lines through (0) as long as the entropy is greater than s_0.

(1) at the back side of the shock. When s has become less than s_0 this situation is reversed. The state (1) will then have greater speed, $u_1^2 + v_1^2 > u_0^2$, and eventually approach the intersection of T_0 with the limit circle, i.e. $u_1^2 + v_1^2 \rightarrow c^2$. A similar discussion could be carried out for $u_0 < c^*$: the state (0) is then of course always on the back side of the shock.

The locus of all points (1) which may be connected with (0) by a shock - or their projections in the (u,v)-plane- is called shock-polar. In case of an ideal gas its equation is easily found. Let the shock direction be given by the angle β with respect to the u-axis. Then we have

(2.10) $$u_{0t} = u_0 \cos\beta ,$$

(2.11) $$u_{0n} = u_0 \sin\beta .$$



For the state (1), omitting the subscript (1), we have from (2.5), (2.6)

(2.12) $$u_t = u_{0t} = u_0 \cos\beta .$$

Dividing (2.7) by (2.6) and eliminating $\rho^{-1}p$ by (1.3), (2.1) and (2.8) one obtains

$$(\gamma - 1)\frac{\hat{c}^2 - u_o^2 - v^2}{u_n} + 2\gamma u_n = (\gamma - 1)\frac{\hat{c}^2 + u_o^2}{u_{on}} + 2\gamma u_{on}$$

or

$$(\gamma - 1)\frac{\hat{c}^2 - u_t^2}{u_n} + (\gamma + 1)u_n = (\gamma - 1)\frac{\hat{c}^2 - u_{ot}^2}{u_{on}} + (\gamma + 1)u_{on}$$

We multiply by $u_r u_{on}$, divide by $u_n - u_{on}$, making use of $u_t^2 = u_{ot}^2$. Then we obtain

(2.13) $$u_{on} u_n = c^{*2} - \frac{\gamma - 1}{\gamma + 1} u_{ot}^2 ,$$

a relation which is analogous to (1.18). From

$$u = u_t \cos\beta + u_n \sin\beta$$

$$v = -u_t \sin\beta + u_n \cos\beta$$

and (2.10), (2.11), (2.12) we obtain the parametric representation

(2.14) $$u = \frac{2}{\gamma + 1} u_o \cos^2\beta + \frac{c^{*2}}{u_o}$$

$$v + [(\frac{2}{\gamma + 1} \cos^2\beta - 1)u_o + c^{*2}/u_o] \cot\beta .$$

Elimination of β yields first

(2.15) $$v = (u - u_o) \cot\beta$$

and then

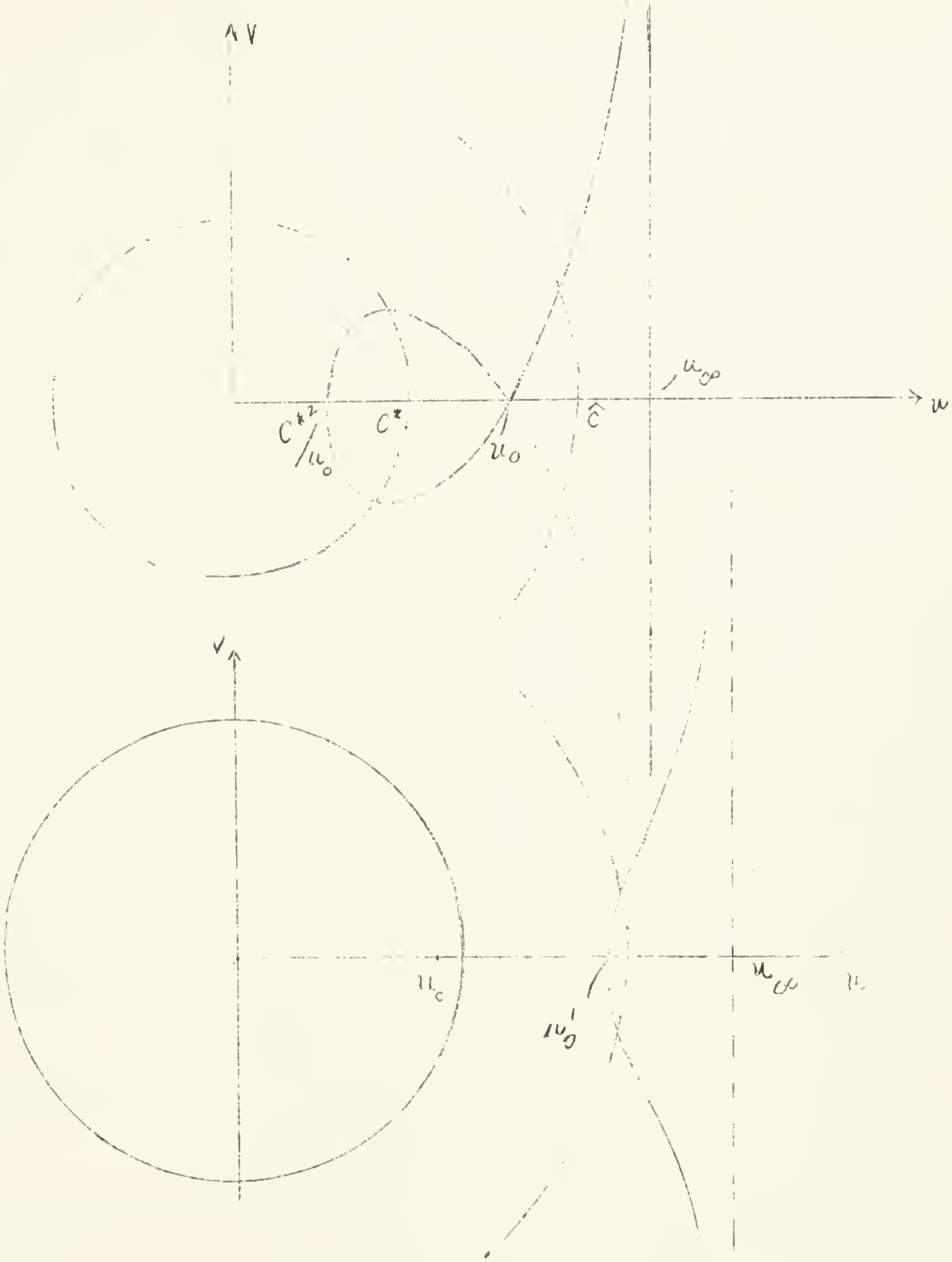
(2.16) $$v^2 = (u - u_o)^2 \frac{u - c^{*2}/u_o}{u_\infty - u}$$

with

(2.17) $$u_\infty = \frac{2u_o}{\gamma + 1} + \frac{c^{*2}}{u_o} .$$

The locus of this equation is a *strophoid* as indicated in the diagram.

Busemann considers anly the "backward" branch for $u_o > c^*$, i.e. the branch consisting of those states (1) which may be connected with (0) in such a way that (0)



is on the front side, (1) on the back side of the shock. The other branches should however not be forgotten.

We observe: When u_0 approaches c^*, the backward branch shrinks to the point $(c^*,0)$. When u_0 approaches $\hat{c}$, the value u also approaches c; the backward branch approaches the circle $v^2 = (\hat{c} - u)(u - c^{*2}/\hat{c})$, while the forward branches approach the line $u = c$.

The backward branch of the shock polar may easily be used to determine the oblique shock when the direction of velocity on the back side is given by the angle α with the u-axis. One need only draw a line in that direction through the origin in the (u,v)-plane and intersect it with the backward shock polar. There will be either two intersections, one intersection, or none at all. If the point of intersection is found on the polar, the shock direction is found as the perpendicular to the line connecting (0) and (1). The normal and tangential components of both velocities are also easily read off from the diagram.

It is interesting to investigate the limit position α_0 of the angle α. This angle depends on the initial state (0); if the initial velocity u_0 approaches c^*, the angle α_0 approaches zero. If, however, the velocity u_0 approaches c, the angle α_0 approaches an "ultimate" value given by

$$\sin \alpha_0 = 1/\gamma \ ;$$

in particular

$$\alpha_0 = 45.5^\circ \quad \text{if} \quad \gamma = 1.4 \ .$$

The corresponding limit angle β_0 for the shock direction is given by

$$\beta_0 = \frac{\pi}{4} + \frac{\alpha_0}{2} \ .$$

The "ultimate" limit value β is therefore

$$\beta_0 = 67.75^\circ \ .$$

It should also be noted that the state corresponding to the limit angle is subsonic.

The angle α under which the polar intersects the

circle $u^2 + v^2 = c^2$ may be denoted by $\hat{\alpha}$; this angle approaches 0 when $u_0 \rightarrow \hat{c}$ or 0 and an ultimate value α^* when $u_0 \rightarrow c^*$.

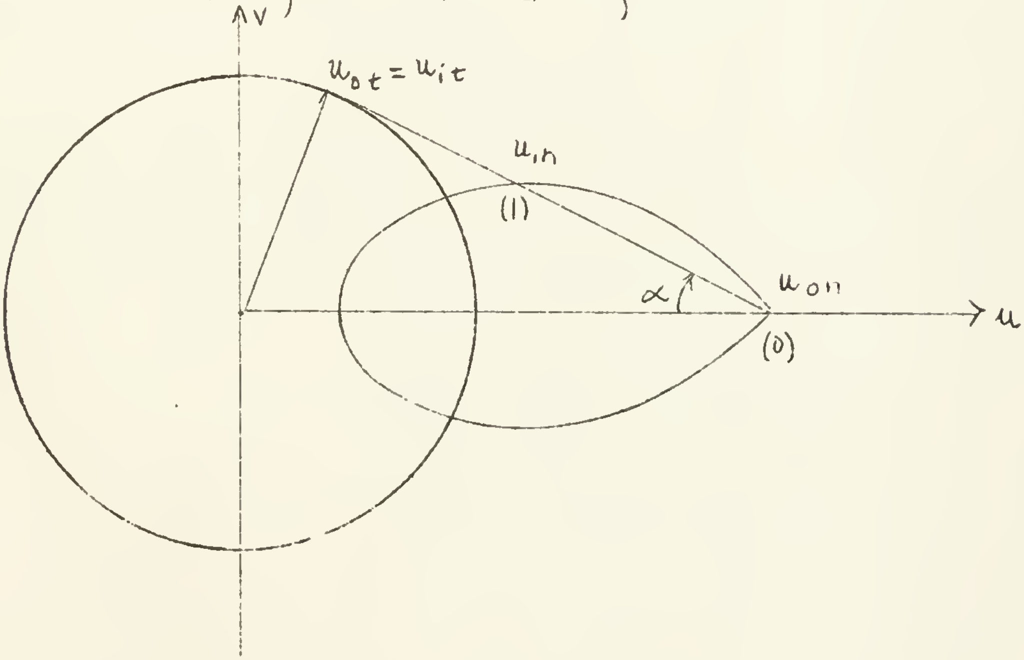
Further: If $\alpha \rightarrow 0$, the state u approaches either to that corresponding to the straight shock or to the state (0) itself. In the latter case the shock direction approaches the characteristic direction which is therefore perpendicular to the tangents at the double point.

It is of course possible to apply, as in one-dimensional case, a Legendre transformation and to consider the two branches of

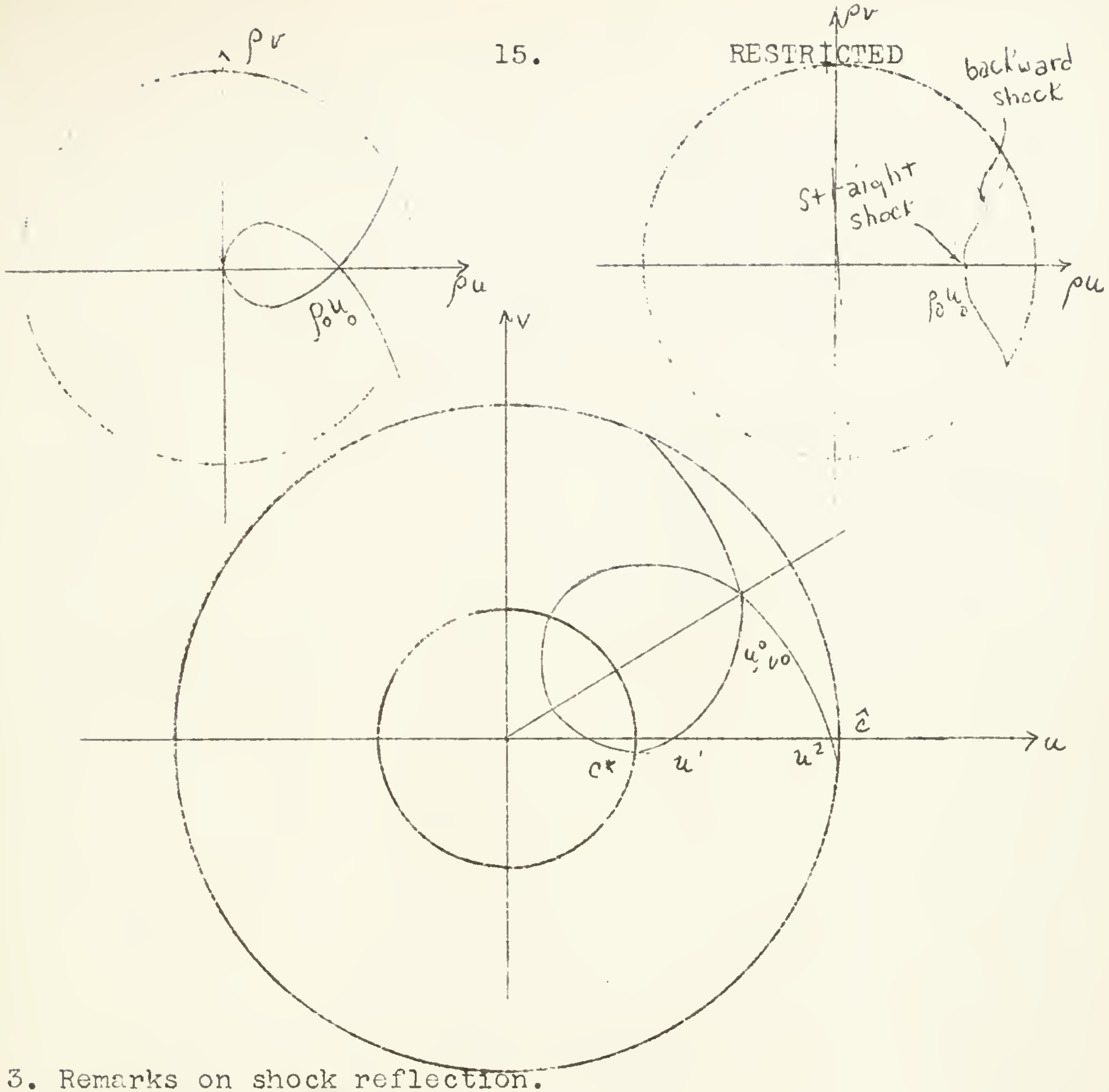
$$q = p + \rho(u^2 + v^2)$$

as functions of ρu and ρv; one then has

$$\partial q/\partial(\rho u) = u, \ \partial q/\partial(\rho v) = v.$$

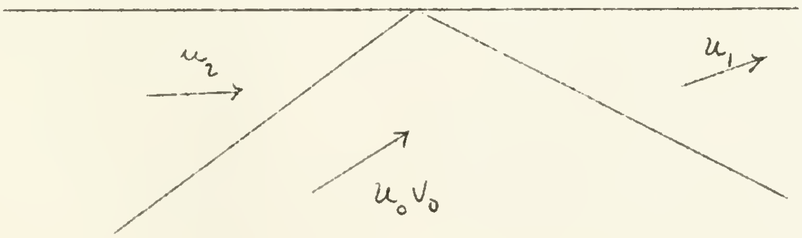


Again the line connecting two points that correspond to both sides of a shock is tangent to both surfaces $q = q(\rho u, \rho v)$. The shock polar connecting a supersonic state with (i) supersonic (ii) subsonic states is indicated in the diagram.



3. Remarks on shock reflection.

Let us consider the steady state flow along a wall crossing two shocks, as indicated in the diagram.

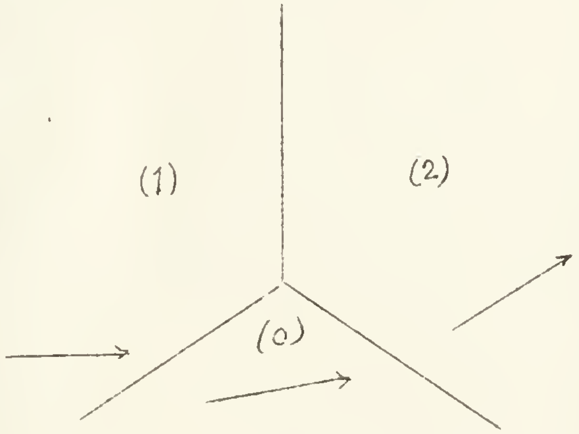


To investigate the possibility of such a flow one may consider (u_o, v_o) and ρ_o, p_o as given, assuming that (u_o, v_o) is supersonic. Then one need only draw the shock polar for the point (u_o, v_o) and observe whether this polar intersects the u-axis in its forward and its backward branch. It is

clear that an intersection with the backward branch will exist only if $v_0/u_0 < \tan \alpha_0$, where α_0 is the limit angle belonging to the state (0). The intersection with the forward branch will exist only if $\alpha < \hat{\alpha}$ (see p.13).

4. Remark on the Mach effect.

Consider the steady flow as indicated in the diagram; the states (1) and (2) are connected by an oblique shock. On the other hand state (1) passes through an oblique shock over to a state (0) and (0) passes over to a state (2) through an oblique shock.



We shall show that such a configuration is impossible. Consider the points (0), (1), (2) in the (u,v,p)-space and the surfaces $p = p_0(u,v)$, $p = p_1(u,v)$, $p = p_2(u,v)$ belonging to them. The lines connecting points (1) and (2), (1) and (0), (0) and (2) are, as was shown, tangent to these surfaces, respectively. The plane spanned by these three points is therefore a tangent plane to all three surfaces. Clearly the intersection of a tangent plane with the (u,v)-plane is perpendicular to the vector

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